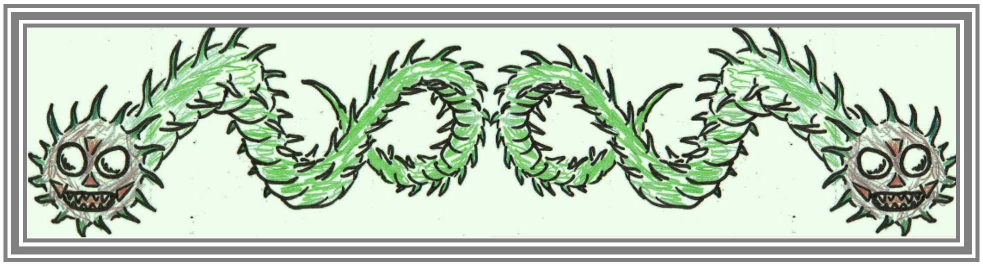
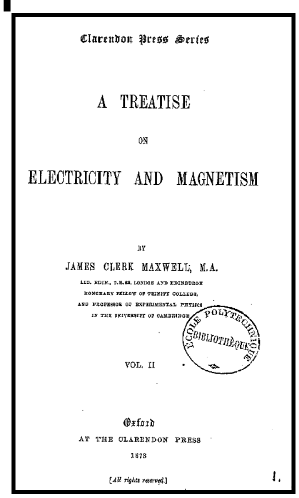


Translation Keys
& Philosophy

Translation of Maxwell’s Equations From Maxwell’s Treatise Part IV Chapters VII to X



Clarendon Press Series

A TREATISE

ON

ELECTRICITY AND MAGNETISM

BY

JAMES CLERK MAXWELL, M.A.

LLD. EDIN., F.R.SS. LONDON AND EDINBURGH
HONORARY FELLOW OF TRINITY COLLEGE,
AND PROFESSOR OF EXPERIMENTAL PHYSICS
IN THE UNIVERSITY OF CAMBRIDGE

VOL. II

Oxford
AT THE CLARENDON PRESS
1873

Rev 0.02

James R. Satterwhite PE Ret.

Teltest Electronics
6/5/2025

Table of Contents

GENERAL EQUATIONS OF THE MAGNETIC FIELD 3

1. INTRODUCTION 3

2. MAXWELL AND QUATERNIONS 6

3. MAXWELL’S VARIABLE NAMES 8

4. INTEGRATION 11

5. DEL, CURL AND SUCH 12

6. MY TRANSLATION NOTATION PRESENTATION 13

7. REFERENCES 14

Table of Figures

FIGURE 1 MAXWELL’S COMMENTS ON QUATERNIONS FROM VOL I OF HIS TREATISE 7
FIGURE 2 SOME OF MAXWELL’S VARIABLE NAMES 9
FIGURE 3 MY ASSIGNMENT OF CORRESPONDING NOTATION 11
FIGURE 2 ILLUSTRATION OF MY CONVERSION PROCESS 14

Tilting Windmills

Volume II Part IV

Chapters VII - X

General Equations of the Magnetic Field

1. Introduction

I have begun this translation of the Equations portion of Maxwell's "A Treatise on Electricity and Magnetism[3]" for a number of reasons. I have desired for years to understand the origins and meaning of Maxwell's Equations, or better stated, Maxwell's Equations and the Maxwell – Heaviside Equations. I have taken a number of academic courses on Engineering Electromagnetics and from what I can see now that was a lot of delving into the fruit of Maxwell's Equations. I have had a long career in Electronic Engineering a good deal of which was successfully dealing in-depth with electromagnetics, yet, understanding Maxwell's Equations eluded me.

Over the last 20 years or so, a number of people have asked me my opinion about a subject that has been bantered about relating to the production of "free energy" that is derived from the application of Maxwell's equations in quaternion form. I had some familiarity with quaternions as I had used them to perform rotations in the data processing of a strap-down inertial platform; however, that had been basically a cookbook implementation and I was really not up-to-date with quaternion algebra. I had read in a number of places that Maxwell in his Treatise had expressed his equations in quaternions. And that, in subsequent editions, at the insistence of his publishers, had removed reference to them.

In any case, my knowledge at the time regarding Maxwell's equations and his Treatise was very limited, almost non-existent and I had to reply that I simply did not know. And, I wanted to know; however, I couldn't find a path to that knowledge, although I did enroll at UT as a doctorial student and took a number of courses over two years in electrical engineering including advanced engineering electromagnetics. Although I could work with the fruit of Maxwell's equations quite well, I really did not understand Maxwell's equations in the way I would like. Over the years I have made a number of stabs at remedying this, including acquiring quite a library on the subject, all to no avail. And then -----

After reading a couple of delightful books on Faraday, Maxwell and Heaviside (See references 1 &2), I decided to dive into it more deeply. I obtained copies of both volumes of his treatise and started my study.

It might help here to explore how Maxwell's Treatise is organized. There are two volumes. Volume II is a continuation of Volume I regarding chapter numbers and article numbers. The article numbers are continuous from the beginning of the Treatise and independent of the Chapter numbering. Also, there is a chapter preceding Chapter I called Preliminary.

My primary interest was in the section of Volume II dealing with the derivations of and presentation of his equations. It was a rude awakening, realizing the difficulty of the task given Maxwell's notation and the limits of conventional notation at the time of his work. The text, for me, is extremely difficult to read. Vector notation, with the exception of quaternions, had not come into common usage at that time. With the aid of vector algebra and its notation, we can express operations on the vector $\vec{B}$ in terms of its components B_x, B_y, B_z or $\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$, or further as some would prefer $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$. Maxwell knew the vector B consisted of three components that he labeled a, b, c and carried out his expression laboriously with each component independently. Not only is this difficult and time-consuming to write, it is a nightmare to read and understand. With modern notation, the component name carries the vector identity, whereas with three different letters for the component names, not related to the vector name; it is taxing to keep track of what is what. This is especially difficult due to the complexity of the material that the reader is attempting to understand.

To add to that, in my view, Maxwell was not diligent in defining his variables prior to using them. (All this makes it more amazing to me that Heaviside, a self taught individual, did read and understand what Maxwell had written, when most of his highly educated, contemporaries could not. Indeed, from this basis he and Gibbs indpendently created the vector algebra and notation in which to express Maxwell's work.) Trying, to read Maxwell's work and keeping track of the variables, and trying to find just what this particular symbol relates to or what he is attempting to present is mind-boggling, to say the least. Some comments from some of Maxwell's students goes to the point here. They loved him dearly and were inspired by him, despite in their opinion, he was a terrible teacher.

A good example is

$$E = \oint_l \left(P\frac{dy}{ds} - Q\frac{dz}{ds} + R\frac{dz}{ds} \right) ds \qquad (5)$$

where

$$\left\{ \begin{aligned} P &= c\frac{dy}{dt} - b\frac{dz}{dt} - \frac{dF}{dt} - \frac{d\psi}{dx} \\ Q &= a\frac{dz}{dt} - c\frac{dx}{dt} - \frac{dG}{dt} - \frac{d\psi}{dy} \\ R &= b\frac{dx}{dt} - a\frac{dy}{dt} - \frac{dH}{dt} - \frac{d\psi}{dz} \end{aligned} \right\} \quad \left| \begin{array}{c} \text{Equations of} \\ \text{Electromotive} \\ \text{Force} \end{array} \right| \qquad (B)$$

$$V = \oint_l \left(E_x\frac{dy}{dl} - E_y\frac{dz}{dl} + E_z\frac{dz}{dl} \right) d\vec{l} \qquad (5)$$

where

$$\left\{ \begin{aligned} E_x &= B_z\frac{dy}{dt} - B_y\frac{dz}{dt} - \frac{dA_x}{dt} - \frac{d\phi}{dx} \\ E_y &= B_x\frac{dz}{dt} - B_z\frac{dx}{dt} - \frac{dA_y}{dt} - \frac{d\phi}{dy} \\ E_z &= B_y\frac{dx}{dt} - B_x\frac{dy}{dt} - \frac{dA_z}{dt} - \frac{d\phi}{dz} \end{aligned} \right\} \quad \left| \begin{array}{c} \text{Equations of} \\ \text{Electromotive} \\ \text{Force} \end{array} \right| \qquad (B)$$

2. Maxwell and Quaternions

Thus, Maxwell was quite limited in his expression of his ideas. Heaviside and Gibbs provided the really needed upgrade in notation, that we engineers take for granted today, particularly vector notation. Not having the benefit of vector algebra and notation, Maxwell was quite burdened in attempting to share his ideas. It seems to me that Maxwell saw the benefit of vector notation in Hamilton's quaternions and latched on to the vector notation in quaternions and as he states in the Preliminary section of Volume I of his Treatise, *On the Relation of Physical Quantities to Directions in Space.*

10.] In distinguishing the kinds of physical quantities, it is of great importance to know how they are related to the directions of those coordinate axes which we usually employ in defining the positions of things. The introduction of coordinate axes into geometry by Des Cartes was one of the greatest steps in mathematical progress, for it reduced the methods of geometry to calculations performed on numerical quantities. The position of a point is made to depend on the length of three lines which are always drawn in determinate directions, and the line joining two points is in like manner considered as the resultant of three lines.

But for many purposes in physical reasoning, as distinguished from calculation, it is desirable to avoid explicitly introducing the Cartesian coordinates, and to fix the mind at once on a point of space instead of its three coordinates, and on the magnitude and direction of a force instead of its three components. This mode of contemplating geometrical and physical quantities is more primitive and more natural than the other, although the ideas connected with it did not receive their full development till Hamilton made the next great step in dealing with space, by the invention of his Calculus of Quaternions.

As the methods of Des Cartes are still the most familiar to students of science, and as they are really the most useful for purposes of calculation, we shall express all our results in the Cartesian form. I am convinced, however, that the introduction of the ideas, as distinguished from the operations and methods of Quaternions, will be of great use to us in the study of all parts of our subject, and especially in electrodynamics, where we have to deal with a number of physical quantities, the relations of which to each other can be expressed far more simply by a few words of Hamilton's, than by the ordinary equations.

Figure 1 Maxwell's Comments on Quaternions From Vol I of His Treatise

I think what Maxwell saw in Hamilton's quaternions was the germ of what vector algebra could be and latched onto Hamilton's vector notation. In my translation to modern notation, I have not seen a single example of his carrying his implementation of quaternions any further than expressions in vector form "calling" them quaternions. When, in fact, he was merely expressing the beginnings of vector notation.

The conclusion I have come to in the process of this translation is that he expressed his equations in vector form and not quaternions even though he thought he was expressing in quaternions. This is not to say that it may be possible to express Maxwell's equations in quaternions or some other advanced algebra to great advantage, I am just saying that I don't believe that Maxwell did. As far as I can tell, Maxwell did not carry out any quaternion algebra or calculation in his Treatise.

3. Maxwell's Variable Names

Maxwell listed some of his variable names in Volume II, Part IV, Chapter VIII, Article 618 of his Treatise. The rest are scattered throughout the Treatise. I will attempt to bring those into my documents as I find them, As stated above, he used a vector name and then three unrelated letters for the variable components of the vector. In addition, he used German script capital letters for the vector names. To me, this was a poor choice, as I found some of the German letters very difficult to distinguish from each other, many times having to reach for my eye loupe to tell which letter was being used. Several of the letters, I have not been able to associate with an English equivalent, and come up with a letter as best as I could from the descriptive name of the variable. All of this makes for very difficult reading of his brilliant ideas. This is the major reason I decided to translate the notation of the section of his Treatise to modern notation for my own edification. Also, my considerable effort in accomplishing this my be useful to other students of Maxwell's Treatise.

Figure 2 below shows his presentation of his variable names and some of the letter assignments I have made.

Quaternion Expressions for the Electromagnetic Equations.

618.] In this treatise we have endeavoured to avoid any process demanding from the reader a knowledge of the Calculus of Quaternions. At the same time we have not scrupled to introduce the idea of a vector when it was necessary to do so. When we have had occasion to denote a vector by a symbol, we have used a German letter, the number of different vectors being so great that Hamilton's favourite symbols would have been exhausted at once. Whenever therefore, a German letter is used it denotes a Hamiltonian vector, and indicates not only its magnitude but its direction. The constituents of a vector are denoted by Roman or Greek letters.

The principal vectors which we have to consider are :—

	Symbol of Vector.		Constituents.		
The radius vector of a point	ρ	$\vec{\Xi}$	x	y	z
The electromagnetic momentum at a point	$\mathfrak{A}$	$\vec{A}$	F	G	H
The magnetic induction	$\mathfrak{B}$	$\vec{B}$	a	b	c
The (total) electric current	$\mathfrak{C}$	$\vec{J}$	u	v	w
The electric displacement	$\mathfrak{D}$	$\vec{D}$	f	g	h
The electromotive force	$\mathfrak{E}$	$\vec{E}$	P	Q	R
The mechanical force	$\mathfrak{F}$	$\vec{F}$	X	Y	Z
The velocity of a point	$\mathfrak{G}$ or $\dot{\rho}$	$\vec{V}$	$\dot{x}$	$\dot{y}$	$\dot{z}$
The magnetic force	$\mathfrak{H}$	$\vec{H}$	α	β	γ
The intensity of magnetization	$\mathfrak{I}$	$\vec{\mathrm{I}}$	A	B	C
The current of conduction	$\mathfrak{K}$	$\vec{j}$	p	q	r

We have also the following scalar functions :—

The electric potential Ψ. ϕ

The magnetic potential (where it exists) Ω. $\wp$

The electric density e. ρ

The density of magnetic 'matter' m. m

Besides these we have the following quantities, indicating physical properties of the medium at each point :—

C, the conductivity for electric currents. σ

K, the dielectric inductive capacity. ε

μ, the magnetic inductive capacity. μ

Rev 4.0

These quantities are, in isotropic media, mere scalar functions of ρ, but in general they are linear and vector operators on the vector functions to which they are applied. K and μ are certainly always self-conjugate, and C is probably so also.

Figure 2 Some of Maxwell's Variable Names

The one character assignment that has given me the most trouble is for the radius vector for a point, he uses ρ; however in modern notation ρ is solidly taken. p would be great; however, that is already taken for momentum. I have, for the time being, selected Ξ, although I am not too satisfied with that selection.

Evidently, the arrow over-stroke for a vector had not come into use at that time, so his vectors receive no distinction that the arrow over-stroke would give. There are times in the text that I can not tell whether or not he intends for the variable to be a vector or a scalar. In my process I have gone through the four chapters Volume II, Part IV, Chapter VII through X, that in my opinion is the essence of his equations expression. This is the first pass. There are areas where I was a bit muddled in the translation. Now with more understanding, I will go back through in more detail, hopefully bringing more clarity to the result. Also, he used very little subscripting. In addition to the Chapters in Volume II relating to equations of electromagnetism, there is information in the Volume I Preliminary Chapter that I have translated as well. Those are Articles 17, 24, 25 and 26.

Figure 3 below shows my version of Maxwell's assignment that he used in the text denoted by braces and my corresponding assignments denoted by brackets.

$$\left\{\vec{\rho} \Leftrightarrow x,y,z\right\} \Leftrightarrow \left[\vec{\Xi} \Leftrightarrow x,y,z\right] ----\text{The radius vector for a point } (meters)$$
$$\left\{\vec{A} \Leftrightarrow F,G,H\right\} \Leftrightarrow \left[\vec{A} \Leftrightarrow A_x,A_y,A_z\right] ----\text{Magnetic Potential } (webers\ /\ meter)$$
$$\left\{\vec{B} \Leftrightarrow a,b,c\right\} \Leftrightarrow \left[\vec{B} \Leftrightarrow B_x,B_y,B_z\right] ----\text{Magnetic Flux Density } (webers\ /\ meter^2)$$
$$\left\{\vec{J} \Leftrightarrow p',q',r'\right\} \Leftrightarrow \left[\vec{J} \Leftrightarrow J_x,J_y,J_z\right] ----\text{Total Current Density } (amperes\ /\ meter^2)$$
$$\left\{\vec{E} \Leftrightarrow P,Q,R\right\} \Leftrightarrow \left[\vec{E} \Leftrightarrow E_x,E_y,E_z\right] ----\text{Electric Field Intensity } (volts\ /\ meter)$$
$$\left\{\vec{D} \Leftrightarrow f,g,h\right\} \Leftrightarrow \left[\vec{D} \Leftrightarrow D_x,D_y,D_z\right] ----\text{Electric Flux Density } (coulombs\ /\ meter^2)$$
$$\left\{\vec{F} \Leftrightarrow X,Y,Z\right\} \Leftrightarrow \left[\vec{F} \Leftrightarrow F_x,F_y,F_z\right] ----\text{Mechanical Force } (neutons)$$
$$\left\{\dot{\vec{\rho}} \Leftrightarrow \dot{x},\dot{y},\dot{z}\right\} \Leftrightarrow \left[\vec{v} \Leftrightarrow v_x,v_y,v_z\right] ----\text{The Velocity of a point } (meters\ /\ \sec)$$
$$\left\{\vec{H} \Leftrightarrow \alpha,\beta,\gamma\right\} \Leftrightarrow \left[\vec{H} \Leftrightarrow H_x,H_y,H_z\right] ----\text{Magnetic Field Intensity } (amperes\ /\ meter)$$
$$\left\{\dddot{I} \Leftrightarrow A,B,C\right\} \Leftrightarrow \left[\vec{I} \Leftrightarrow I_x,I_y,I_z\right] ----\text{Intensity of Magnetization (Units ???) ???}$$
$$\left\{\dddot{N} \Leftrightarrow p,q,r\right\} \Leftrightarrow \left[\vec{j} \Leftrightarrow j_x,j_y,j_z\right] ----\text{Conduction Current Density } (amperes\ /\ meter^2)$$

$$\left\{\Psi\right\} \Leftrightarrow \left[\Phi\right] ----\text{Electric Potential (?)}$$
$$\left\{\Omega\right\} \Leftrightarrow \left[\wp\right] ----\text{Magnetic Potential (?)}$$
$$\left\{e\right\} \Leftrightarrow \left[\rho\right] ----\text{Electric Charge Density } (coulombs\ /\ meter^2)$$
$$\left\{m\right\} \Leftrightarrow \left[m\right] ----\text{Density of Magnetic 'matter' (?)}$$

$$\left\{C\right\} \Leftrightarrow \left[\sigma\right] ----\text{Conductivity for Electric Currents (?)}$$
$$\left\{K\right\} \Leftrightarrow \left[\varepsilon\right] ----- \text{ Dielectric Inductive Capacity ()}$$
$$\left\{\mu\right\} \Leftrightarrow \left[\mu\right] ----- \text{ Magnetic Inductive Capacity ()}$$

Rev 5.0

Figure 3 My Assignment of Corresponding Notation

4. Integration

Maxwell used the variable s for the line-integral and S for the surface-integral. Modern usage would employ $\vec{l}$ for the line-integral. Also, for the closed line and surface-integrals he did not employ the circle on the integral sign.

$$\int f\ ds \quad \Rightarrow \oint_l f\ d\vec{l}$$

$$\int f\ dS \quad \Rightarrow \oint_S f\ d\vec{S}$$

5. Del, Curl and Such

Although he uses and defines the Del operator ∇ in Article 619, he did not have access to the definitions and notation to come later in the form of the Dot Product, the Cross Product, the Divergence or the Curl. For example in Article 619, he develops the component equations for the curl and then has to express it in a way that is not clear. It must have been painful to develop the concept so fully and have no adequate way to express it as a vector. He did come up with a way to hint at what he was thinking,

619.] The equations (A) of magnetic induction, of which the first is,

$$a = \frac{dH}{dy} - \frac{dG}{dz},$$

may now be written $\mathfrak{B} = V \nabla \mathfrak{A}$,

where ∇ is the operator

$$i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz},$$

and V indicates that the vector part of the result of this operation is to be taken.

$$\left\{\begin{aligned} B_x &= \frac{dA_z}{dy} - \frac{dA_y}{dz}, \\ B_y &= \frac{dA_x}{dz} - \frac{dA_z}{dx}, \\ B_z &= \frac{dA_y}{dx} - \frac{dA_x}{dy}, \end{aligned}\right\} \qquad (A)$$

$$\boxed{\vec{B} = V\nabla\vec{A} = \nabla \times \vec{A}}$$

He writes that B is the vector part of ∇A,

$B = V\nabla A$.

Expanding equation (A) to vector form,

$$\vec{B}=\left(\frac{dA_z}{dy}-\frac{dA_y}{dz}\right)\hat{x}+\left(\frac{dA_x}{dz}-\frac{dA_z}{dx}\right)\hat{y}+\left(\frac{dA_y}{dx}-\frac{dA_x}{dy}\right)\hat{z},$$

or

$$\vec{B}=\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\hat{x}+\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\hat{y}+\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\hat{z},\ \text{it is}$$

$$\boxed{\vec{B}=\vec{\nabla}\times\vec{A}},$$

obvious that he has clearly derived the formula for the curl of $\vec{A}$; however, has no notation for it.

6. My Translation Notation Presentation

Above, there have been hints at how I will present my notation translation of Maxwell's Treatise Equations derivation. I will take an example and explain. The selected example is from Vol II, Part IV, Chapter VII "Theory of Electric Circuits". First, I insert a manageable snip from the pdf of the Treatise. Below that I type the text as Maxwell wrote it along with his notation. His notation is followed by my version of his notation in modern terms in blue font. The intent on this first pass is to allow the reader to see Maxwell's own words and equations and how I changed the equations into modern form. At some point I may remove the snip and presentation of his formulas.

Let T denote the electrokinetic energy of the system. It is a homogeneous function of the second degree with respect to the strengths of the currents, and is of the form

$$T=\tfrac{1}{2}L_1\dot{y}_1{}^2+\tfrac{1}{2}L_2\dot{y}_2{}^2+\&c.+M_{12}\dot{y}_1\dot{y}_2+\&c., \qquad (1)$$

where the coefficients L, M, &c. are functions of the geometrical variables x_1, x_2, &c. The electrical variables y_1, y_2 do not enter into the expression.

Let $T\left(\widehat{E}\right)$ denote the electrokinetic energy of the system. It is a homogenous function of the second degree with respect to the strengths of the currents, and is of the form

$$T=\frac{1}{2}L_1\dot{y}_1{}^2+\frac{1}{2}L_2\dot{y}_2{}^2+M\dot{y}_1\dot{y}_2+\&c., \qquad (1)$$

$$\Updownarrow$$

$$\widehat{E}=\frac{1}{2}L_1i_1{}^2+\frac{1}{2}L_2i_2{}^2+M_{12}i_1i_2+\&c., \quad joules \qquad (S1)$$

Figure 4 Illustration of My Conversion Process

(Maxwell uses T here to represent energy. Since the symbol normally used for energy is E and that conflicts with E for the electric field, I have chosen to use $\widehat{E}$ here to represent energy.)

where the coefficients L_1, L_2, & M_{12} are functions of the geometrical variables x_1, x_2 & c. The electrical variables $\mathrm{y}_1\left(\int i_1 dt\right)$, $\mathrm{y}_1\left(\int i_2 dt\right)$ do not enter into the expression.

7. References

1. "Faraday, Maxwell and the Electromagnetic Field" by Nancy Forbes and Basil Mahon
2. "The Forgotten Genius of Oliver Heaviside" by Basil Mahon
3. ""A Treatise On Electricity & Magnetism", Volumes I & II, by James Clerk Maxwell
4. "The Dynamical Theory of the Electromagnetic Field" by James Clerk Maxwell
5. "A Students Guide to Maxwell's Equations" by Daniel Fleisch Edited by Thomas F. Torrance
6. "Advanced Engineering Electromagnetics" by Constantine A. Balanis
7. "Electromagnetics for Engineers" by Clayton R. Paul
8. "Ordinary Differential Equations" by Morris Tenenbaum and Harry Pollard
9. "Partial Differential Equations for Scientists and Engineera" by Stanley J. Farlow
10. "Quaternions and Rotational Sequences" by Jack B. Kuipers
11. "Advanced Calculus for Applications" Francis B. Hilderbrand
12. "Advanced Engineering Mathematics" by C. R. Wylie, Jr.
13. "Electromagnetic Theory for Engineering Applications" by Walter L. Weeks
14. "Operation Mathematics" by Ruel V. Churchill
15. "Functions of a Complex Variable and some of their applications" byB. A. Fuchs and B. V. Shabat translated from Russian by J. Berry and revised and expanded by J. W. Reed
16. "div, grad, curl and all that, an informal text on vector calculus" fourth edition by H. M. Schey

End of Document

D:\!\Maxwell\Translation\TransKey\TELIMaxwellTranlationKey-R_2.doc